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Homogenization procedure and Pade approximations in the theory of composite materials with parallelepiped inclusions

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Abstract—An analytical solution, describing homogenized coefficients for composite materials with periodic cylindrical inclusions of square section domain, has been obtained by asymptotic methods and Pade approximants for any size of inclusions and its conductivity. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

One of the main tasks of the theory of dispersed media is a theoretical prediction of the effective transport properties. The subject we wish to discuss in this paper has a long history, dating back to Maxwell [2]. The problem may be formulated in a number of mathematically equivalent ways, but here we shall discuss it in the language of heat conductivity. We wish to determine the effective conductivity λ of an infinite simple cubic lattice of identical macroscopic cubic inclusions (holes in particular case), immersed in a matrix. In ref. [1] is displayed a table showing light distinct physical problems, which may be solved by analogous mathematical methods. One of these is the above-mentioned elastic constant, while other involve calculating the dielectric and constants, the magnetic permeability, and electric conductivity, etc.

The calculation of λ for a general type of composite was originally discussed by Maxwell [2], who considered each particle of the composite as an isolated dipole. The second-order approximation was due to Rayleigh [3], who took into account particle moments up to the octupole and calculated the effective transport coefficients from a truncated system of linear algebraic equations. An historical review of the subject, including an exhaustive list of references, was compiled in refs [1, 4].

In the 1970s composite materials with regularly spaced cylindrical inclusions became the subject of interest [5–8] owing to their new applications, such as absorbers of solar energy [5]. Composite materials with square or rectangular fibers were studied in refs [9–11]. In refs [9] and [10], methods of nets and finite elements were used, respectively. In ref. [11], a limiting case for large (close to the maximal value) rectangular cross-section cylindrical cavities was studied by asymptotic procedure, and simple analytical expressions for effective parameters were produced. It is worth noting that numerical methods, in some cases, give satisfied solutions [9], but, in general, its use is not simple.

In much research the so-called three-phase model (TPM) is used [12–14]. Due to this approach we replace all periodic structures, with the exception of one cell, by homogenized media with unknown characteristics (Fig. 1). From the mathematical point of view it leads to the replacement of the periodicity conditions to the conditions of junction of cell with homogenized media. Then we come to the problem of two-phase inclusions in the infinite domain. It gives the possibility of using the method of boundary form perturbations, replaced in the first approximation contour of any inclusion on the spherical one.

The power-series expansion has been used to construct Pade-approximants and continued fraction representation of the effective conductivity in ref. [15]. Continued fractions are successfully used in the theory of composite materials as for calculations of effective constants and for estimation of upper and lower

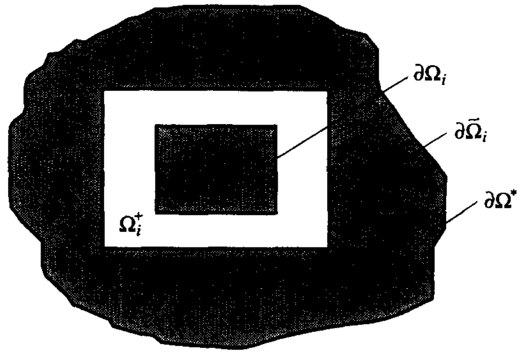

Fig. 1. Three-phase model of the periodic structures.

bounds for it. Neither of the above-mentioned methods yield accurate results for a system of perfectly conductive, nearly touching spheres. In order to describe such a system, in ref. [8], we have derived an asymptotic formula. However, the validity range of this formula is not known. Moreover, there still remains a certain parameter range which is covered neither by the asymptotic formula nor by the solutions based assumption of small concentration of inclusions.

As shown in ref. [16], two-point Pade approximants (TPPA) may be effectively used for the study of the effective heat conductivity λ of a periodic square array of nearly touching cylinders of the conductivity embedded in a matrix material of the conductivity. A sequence of TPPA was constructed for the effective conductivity of the system. The TPPA form a sequence of rapidly converging upper and lower bounds on the effective conductivity.

This paper aims to predict the effective conductivity of two-phase material, which consists of a regular array of parallelepipeds embedded in a homogeneous continuous phase referred to as a matrix, and is organized as follows. In Section 2 we describe the homogenization procedure. TPM is used for the solving of the so-called local problem in Section 3. A description of the asymptotic procedure for the case of large inclusions is given in Section 4. In Section 5 we use Pade approximants to obtain analytical expressions for an effective parameter, containing all asymptotics for any limiting value of volume and rigidity. In Section 6, we present two-sides estimation. Section 7 discusses the advantages and limitations of our method in the light of the results from the previous sections.

2. GOVERNING RELATIONS AND HOMOGENIZATION PROCEDURE

We study the effective parameters of a periodic array (period 2) of cylinders with square cross-section (side length $2a$) embedded in a matrix material. Governing relations may be written as follows:

$$\Delta u^+ = f \quad \text{in } \Omega^+ \quad \Delta u^- = f \quad \text{in } \Omega^- \tag{1}$$

$$u^+ = u^- \quad \frac{\partial u^+}{\partial n} = \lambda \frac{\partial u^-}{\partial n} \quad \text{on } \partial\Omega_i \tag{2}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{3}$$

Here, the indices '+' and '−' denote the matrix (Ω^+) and elastic inclusion (Ω^-); $\lambda = G^-/G^+$; n is the outer normal to the contour of inclusion $\partial\Omega_i$; $\partial\Omega$ is the boundary of domain Ω ($\Omega = \Omega^+ \cup \Omega^-$, typical size $2L$). We also denote $\varepsilon = 1/L$ ($\varepsilon \ll 1$); $\xi = \varepsilon^{-1}x$, $\eta = \varepsilon^{-1}y$.

The study of such problems is important from a theoretical, as well as a numerical, point of view. Because of the complicated structure of the multiply-connected domain, any kind of calculation is difficult to perform. It we treat the boundary value problem we have to impose the boundary condition on the boundary of the inclusions, which are numerous. So, we would like to approximate the given problem by a homogenized problem on the domain without inclusions. By the method of asymptotic development, a problem on a periodically perforated domain is reduced to solving problems in the 'basic cell' and in the domain without holes.

The theory of homogenization has been developed by many authors [11, 17] (we refer to these publications just quoted for bibliographical references). The main problem in this field is in the solving of the so-called cell (or local) problem. This problem has usually been treated by the numerical method. We have used TPM, singular perturbation and TPPA for solving the cell problem and have constructed an approach in this paper.

The operators $\partial/\partial x$ and $\partial/\partial y$ applied to functions $u^{\pm}$ become

$$\partial/\partial x = \partial/\partial x + \varepsilon^{-1}\partial/\partial\xi \quad \partial/\partial y = \partial/\partial y + \varepsilon^{-1}\partial/\partial\eta. \tag{4}$$

Let us represent the solution in the form of a formal expansion

$$u^{\pm} = u_0(x, y) + \varepsilon u_1^{\pm}(x, y, \xi, \eta) + \varepsilon^2 u_2^{\pm}(x, y, \xi, \eta) + \cdots. \tag{5}$$

Substituting series (5) into boundary value problems (1)–(3), taking into account relations (4) and splitting it in respect to the powers of ε, anyone can obtain the recurrent sequence of boundary value problems. The local problem is formulated for the cell as follows:

$$\Delta u_1^+ = 0 \quad \text{in } \Omega_i^+ \quad \Delta u_1^- = 0 \quad \text{in } \Omega_i^-$$

$$u_1^+ = u_1^- \quad \frac{\partial u_1^+}{\partial \bar{n}} - \lambda \frac{\partial u_1^-}{\partial \bar{n}} = (\lambda - 1)\frac{\partial u_0}{\partial n} \quad \text{on } \partial\Omega_i \tag{6}$$

$$u_1^+|_{\xi=1} = u_1^+|_{\xi=-1} \quad \left.\frac{\partial u_1^+}{\partial \xi}\right|_{\xi=1} = \left.\frac{\partial u_1^+}{\partial \xi}\right|_{\xi=-1} \quad (\xi \rightleftarrows \eta) \tag{7}$$

$\bar{n}$ is the outer normal in fast variables.

3. TPM SOLUTION OF CELL PROBLEM

For the solving of the local problem we use TPM. Then we come to the problem of two-phase inclusions in the infinite domain. Using the method of boundary form perturbations, we replace in the first approximation a square contour on the circle one and come to the following boundary value problem

$$\Delta u_1^+ = 0 \quad \text{in } \Omega_i^+ \quad \Delta u_1^- = 0 \quad \text{in } \Omega_i^-$$

$$\Delta \tilde{u}_1 = 0 \quad \text{in } \tilde{\Omega} \tag{8}$$

$$u_1^+ = u_1^- \quad \text{for } r = 2a\sqrt{\pi} \tag{9}$$

$$\frac{\partial u_1^+}{\partial r} - \lambda \frac{\partial u_1^-}{\partial r} = (\lambda - 1)\left[\frac{\partial u_0}{\partial x}\cos\theta + \frac{u_0}{\partial y}\sin\theta\right]$$

$$[u_1^- \to \tilde{u}_1\,;\quad \lambda \to \tilde{\lambda}] \quad \text{for } r = 2\sqrt{\pi} \tag{10}$$

$$\tilde{u}_1 \to 0\,;\quad \frac{\partial \tilde{u}_1}{\partial r} \to 0 \quad \text{for } r \to \infty. \tag{11}$$

The solution of (8)–(11) may be represented in the form

$$u_1^- = A_1 r\cos\theta + A_2 r\sin\theta$$

$$\tilde{u}_1 = D_1/r\cos\theta + D_2/r\sin\theta$$

$$u_1^+ = [B_1 r + C_1/r]\cos\theta + [B_2 r + C_2/r]\sin\theta. \tag{12}$$

Here

$$A_1 = -[1 + 4\tilde{\lambda}\Lambda]\frac{\partial u_0}{\partial x}$$

$$B_1 = -[1 + 2(\lambda+1)\tilde{\lambda}\Lambda]\frac{\partial u_0}{\partial x}$$

$$C_1 = 8/\pi a^2(\lambda - 1)\tilde{\lambda}\Lambda\frac{\partial u_0}{\partial x}$$

$$D_1 = 4/\pi[1 + 2[a^2(\lambda-1) + \lambda + 1]\Lambda]\frac{\partial u_0}{\partial x}$$

$$\Lambda = [[\tilde{\lambda} - 1][\lambda + 1]a^2 - [\tilde{\lambda} + 1][\lambda + 1]]^{-1}$$

$$A_2 = A_1 \quad B_2 = B_1 \quad C_2 = C_1$$

$$D_2 = D_1 \quad \left[\frac{\partial u_0}{\partial x} \rightleftarrows \frac{\partial u_0}{\partial y}\right].$$

For the determination of an effective parameter of homogenized media we use the equation

$$\Phi[u_1^+] + \lambda\Phi[u_1^-] + \tilde{\lambda}\Phi[\tilde{u}_1] = f$$

where $\Phi(u) = \Delta_{xy}u_0 + 2\tilde{\Delta}u_1 + \Delta_{\xi\eta}u_2$; Δ_{xy}, $\Delta_{\xi\eta}$ are the Laplace operators in 'slow' and 'fast' variables; $\tilde{\Delta} = \partial^2/\partial x\partial\xi + \partial^2/\partial y\partial\eta$; and the operator of averaging is:

$$(\tilde{\ldots}) = \frac{1}{|\Omega^*|}\left[\iint_{\Omega_i^+}\Phi(\ldots)\,d\xi\,d\eta + \lambda\iint_{\Omega_i^-}\Phi(\ldots)\,d\xi\,d\eta + \tilde{\lambda}\iint_{\tilde{\Omega}}\Phi(\ldots)\,d\xi\,d\eta\right]$$

where $\Omega^* = \Omega_i^+ \cup \Omega_i^- \cup \tilde{\Omega}$.

Then the homogenized equation may be written in the form

$$\Delta u_0 + \frac{1}{|\Omega^*|}\left[\iint_{\Omega_i^+}\tilde{\Delta}u_1^+\,d\xi\,d\eta + \lambda\iint_{\Omega_i^-}\tilde{\Delta}u_1^-\,d\xi\,d\eta + \tilde{\lambda}\iint_{\tilde{\Omega}}\tilde{\Delta}\tilde{u}_1\,d\xi\,d\eta\right] = f$$

and an unknown effective parameter may be obtained from the linear algebraic equation as follows

$$\tilde{\lambda} = q = \frac{\lambda(1+a^2) + 1 - a^2}{\lambda(1-a^2) + 1 + a^2}. \tag{13}$$

Let us consider various asymptotic expressions arising from equation (13).

(1) $a \to 0$. for any value of λ we have

$$q = 1 + 2(\lambda-1)/(\lambda+1)a^2 \to 1.$$

In other words, in limit we have homogeneous media with an effective parameter equal to the matrix characteristics. In particular:

(2) $\lambda \to 0$ (inclusions of small conductivity, perforation in limit)

$$q = 1 - 2a^2 + 4\lambda a^2.$$

(3) $\lambda \to \infty$, i.e. for inclusions of large conductivity

$$q = 1 + 2a^2 - 4a^2/\lambda.$$

(4) $a \to 1$. For any value of λ we have

$$q = \lambda + (1-a)(1-\lambda^2) \to \lambda$$

i.e. in limit we have an homogeneous media with an effective parameter equal to the inclusion characteristics. In particular:

(5) $\lambda \to 0$ (inclusions of small conductivity)

$$q = 1 - a + \lambda.$$

(6) $1 \ll \lambda \ll 1/(1-a)$ (inclusions of large but finite conductivity)

$$q = \lambda(1 - \lambda(1-a)).$$

(7) $\lambda \to 0$—inclusions of small conductivity, perforation in limit. Then

$$q = (1-a^2)/(1+a^2) + 4a^2\lambda/(1+a^2)^2.$$

(8) $a \ll 1$—for small holes

$$q = 1-2a^2+4a^2\lambda.$$

(9) $a \to 1$—for large inclusions

$$q = 1-a+\lambda.$$

It coincides with known results for perforations [11].

(10) $\lambda \to 1$; the parameters of the matrix and inclusions differ slightly

$$q = 1+(\lambda-1)a^2 \to 1$$

i.e. in limits we have homogeneous media.

(11) $\lambda \to \infty$. In this case, the conductivity of inclusions tends to infinity.

$$q = (1+a^2)/(1-a^2)-4a^2/(\lambda(1-a^2)^2)$$

(12) For $a \ll 1$

$$q = 1+2a^2-4a^2/\lambda.$$

(13) For $a \to 1$ and $1 \ll 1/(1-a) \ll \lambda$

$$q = 1/(1-a)-1.$$

The comparison of known perturbation [11] and numerical [10] results with our calculations confirms the great accuracy of the approach proposed.

4. SOLUTION OF CELL PROBLEM FOR LARGE INCLUSIONS

Let us now consider $a \to 1$ (large inclusions, Fig. 2). Here we cannot use the previous approach, but the smallness of the parameter thickness of the wall between two holes (see Fig. 2), may be taken into account. Then we may construct an asymptotic solution, using singular perturbation technique, similar to that proposed in ref. [18]. Thanks to the symmetry, we may consider each strip (see Fig. 2), separately, and obtain a solution for only one of them, for example, Ω_{i1}^+. For this strip it may be easily shown that $u_{i1\xi\xi}^+$ may be neglected in comparison with $u_{1\eta\eta}^+$

$$\frac{\partial^2 u_{11}^+}{\partial \eta^2} = 0 \quad \text{in } \Omega_{i1}^+ \quad \lambda \Delta u_1^- = 0 \quad \text{in } \Omega_i^-$$

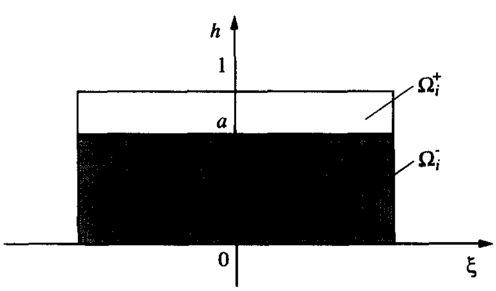

Fig. 2. Model of structures with large inclusions.

$$u_1^+ = u_1^- \quad \frac{\partial u_{11}^+}{\partial \eta} - \lambda \frac{\partial u_1^-}{\partial \eta} = (\lambda-1)\frac{\partial u_0}{\partial y}$$

$$\text{for } \eta = a. \quad (14)$$

For the condition of periodic continuation, let us represent it as follows:

$$u_{11}^+ = 0 \quad \text{for } \eta = a. \quad (15)$$

The solution of boundary value problem (14)–(15) is

$$u_{11}^+ = A_1 + B_1\eta \quad u_1^- = D_1\eta \quad (16)$$

where

$$A_1 = [1-\lambda\Lambda]\frac{\partial u_0}{\partial y} \quad B_1 = -[1-\lambda\Lambda]\frac{\partial u_0}{\partial y}$$

$$D_1 = -[1-\Lambda]\frac{\partial u_0}{\partial y} \quad \Lambda = [a+\lambda(1-a)]^{-1}.$$

The effective parameter, q, we obtain from the homogenized equation

$$q = \frac{\lambda(1-a^2+a^3)+a^2(1-a)}{\lambda(1-a)+a}. \quad (17)$$

In the various limiting case we have:

(1) $a \to 1$, then for any value of λ we have

$$q = \lambda + (1-a)(1-\lambda^2) \to \lambda.$$

This result coincides with the TPM approach.

(2) $\lambda \to 0$

$$q = a(1-a)+\lambda(1-a+a^2)/a$$

that corresponds in the limiting case to the perforated media.

(3) $a \to 1$

$$q = 1-a+\lambda$$

it coincides with the prediction of TPM and for the case of perforations [11].

(4) $\lambda \to 1$, for any value of a we have

$$q = 1+a(\lambda-1)(1-a+a^2) \to 1.$$

(5) $\lambda \to \infty$, inclusions of infinite conductivity,

$$q = (1-a^2+a^3)/(1-a)-a(1-a+a^2)/((1-a)^2\lambda).$$

(6) $a \to 1$ and $1 \ll 1/(1-a) \ll \lambda$,

$$q = 1/(1-a)-1.$$

These asymptotics coincide with the prediction of TPM. Thus, we may conclude that TPM predicts correctly all limiting values of the effective parameter.

5. THREE- AND TWO-POINT PADE-APPROXIMANTS

Practically any physical or mechanical problem, whose parameters include the variable parameter ε, can be approximately solved as it approaches zero or infinity. How can this 'limiting' information be used in the study of a system at the intermittent values of ε? This problem is one of the most complicated in the asymptotic analysis. In many instances the answer to it is alleviated by many-point Pade approximants [19].

The notion of three-point Pade approximants is defined by Baker and Graves-Morris [19]. Let

$$F(\varepsilon) = \sum_{i=0}^{\infty} a_i \varepsilon^i \quad \text{when } \varepsilon \to 0 \tag{18}$$

$$F(\varepsilon) = \sum_{i=0}^{\infty} b_i(-1+\varepsilon)^i \quad \text{when } \varepsilon \to 1 \tag{19}$$

$$F(\varepsilon) = \sum_{i=0}^{\infty} c_i \varepsilon^{-i} \quad \text{when } \varepsilon \to \infty. \tag{20}$$

Three-point Pade approximants represented by the function

$$F(\varepsilon) = \left(\sum_{i=0}^{m} \alpha_k \varepsilon^k\right) \Big/ \left(\sum_{i=0}^{n} \beta_k \varepsilon^k\right)$$

in which k_1, k_2 and k_3 $(k_1+k_2+k_3 = m+n+1)$ coefficients of expansion in the Taylor series when $\varepsilon \to a$, $\varepsilon \to b$ and $\varepsilon \to c$) coincide with the corresponding coefficients of the series (18), (19) and (20), respectively.

Below we will use, for $a \to 0$, the well-known Voight–Reiss estimations [14]

$$q = 1+a^2 \quad \text{for } \lambda \to \infty \tag{21}$$

$$q = 1-a^2 \quad \text{for } \lambda \to 0 \tag{22}$$

and evident relation

$$q = 1 \quad \text{for } \lambda = 1. \tag{23}$$

Let us match expressions (21)–(23) due to variable λ, using three-point Pade approximants

$$q = \frac{1-a^2+\lambda+\lambda^2}{1+\lambda+\lambda^2(1-a^2)}. \tag{24}$$

For $a \to 1$ we use the asymptotic formula (17).

Then we match expressions (17) and (24), due to variable a, by two-point Pade approximants and obtain

$$q = \frac{(1+\lambda+\lambda^2)-a(1+\lambda^2)+a^2\lambda^2}{(1+\lambda+\lambda^2)-a(1+\lambda^2)+a^2}. \tag{25}$$

From the formula (25) all above-mentioned limiting asymptotics for any value of a and λ may be obtained. The values of coefficient q (25) are in good agreement with the results obtained by TPM. So, the comparison of homogenized coefficients due to the size of inclusions (with fixed value of L) shows maximal discrepancies (not exceeding 10%) for some intermediate value between 'small' and 'large' size—$a \simeq 0.5$. This discrepancy decreases with decreasing difference between the conductivity of the matrix and the inclusions. For a fixed size of inclusion maximal discrepancies (not exceeding 10%) are in limiting case—cavities and inclusions of infinite conductivity—and for $a \simeq 0.5$.

6. TWO-SIDES ESTIMATIONS

Two-sides estimations obtained by Hashin and Shtrikman are widely used in the theory of composite materials [14]. It is very useful in many cases, but, unfortunately, it is not when taking into account the inclusions form, and in limiting for conductivity cases ($\lambda \to 0$ and $\lambda \to \infty$) gives trivial bounds. As will be shown below, Pade approximants matching give the possibility of estimating effective parameters for given forms of inclusions for any value of size and conductivity.

As a governing relation for an effective parameter we chose TPM solution (13). For any value of a, the formula (25) gives upper bound of TPM for $\lambda < 1$ and low bound for $\lambda > 1$. Let us obtain low bounds of TPM for $\lambda < 1$ and upper bounds for $\lambda > 1$.

Let us suppose that inclusion small ($a \to 0$) and to use solution for small circle inclusion, obtained in ref. [20] by Schwarz approach [21] (two approximation taking into account). This solution gives low bound for small conductivity of inclusions and upper, for large. Now we will suppose that the radius of inclusion is the function of variables, i.e. $a = a(\xi, \eta)$. Then the solution of the local problem is

$$u_1^- = -(\lambda-1)/(\lambda+1)\left(\frac{\partial u_0}{\partial x}\xi + \frac{\partial u_0}{\partial y}\eta\right)$$

$$u_1^+ = u_{11}^+ + u_{12}^+$$

$$u_{11}^+ = -(\lambda-1)/(\lambda+1)[\xi/(\xi^2+\eta^2) - 0.25\pi\xi](a^2+\eta^2)\frac{\partial u_0}{\partial x}$$

$$u_{11}^+ = u_{12}^+(\xi \rightleftarrows \eta; \ \ x \rightleftarrows y). \tag{26}$$

After the use of routing homogenization procedure, one obtains an expression for the homogenized coefficient

$$q = 1+(\lambda-1)(\lambda+1)a^2(1+\pi/4+\pi/6a-5\pi/12a^2). \tag{27}$$

Matching due to the size of the inclusion on asymptotic expression for large inclusion (17), with obtaining for small inclusions solution (27) (for this purpose we use two-point Pade approximants), we have

$$q = \frac{4(1+\lambda)-4(1+\lambda)a+\lambda(4+\pi)a^2-\lambda\pi a^3}{4(1+\lambda)-4(1+\lambda)a+(4+\pi)a^2-\pi a^3}. \tag{28}$$

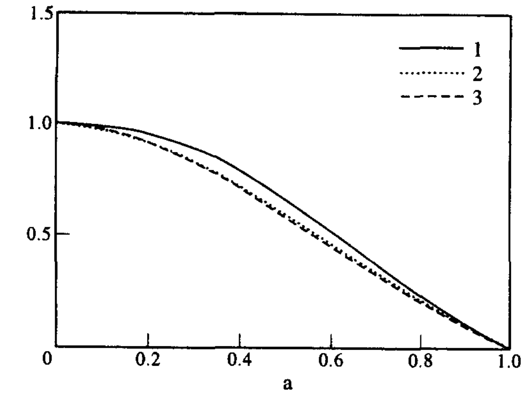

Fig. 3. Dependence of homogenized parameter due to size of inclusions for small inclusion conductivity $\lambda = 0.0001$.

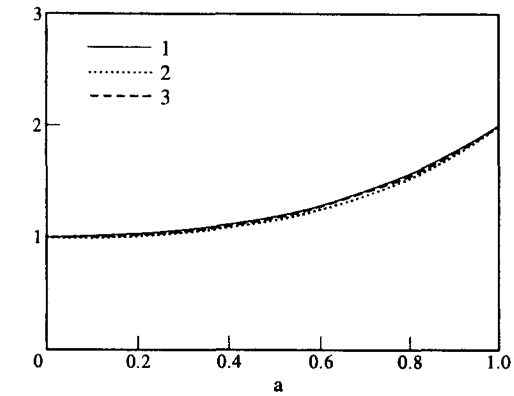

Fig. 4. Dependence of homogenized parameter due to size of inclusions for middle inclusion conductivity $\lambda = 2$.

Formula (28) gives low bounds of TMP for $\lambda < 1$ and upper, *f* for $\lambda > 1$.

Simultaneously using expressions (25) and (28) gives two-sides estimations for an effective parameter for any value of size inclusion and its conductivity, and the discrepancy between it does not exceed 10%, even in limiting cases ($\lambda \to 0$; $\lambda \to \infty$ and $a \cong 0.5$).

7. NUMERICAL RESULTS

Some numerical results are shown in Figs. 3–8, where the effective conductivity values obtained are due to TPM marked as curve 1 and obtained by formula (25) and (27) as curve 2,3, respectively. In Figs. 3–5, the influence of inclusions size is analyzed. The value of λ equalling 0.0001, corresponds to the composite material with cavities, $\lambda = 2$ is the conductivity of the matrix and inclusions of the same order, $\lambda = 100$ are the inclusions of infinite conductivity. In Figs. 6–8, the influence of inclusions conductivity is analyzed for its various sizes: small ($a = 0.25$), middle ($a = 0.5$) and large ($a = 0.75$).

8. CONCLUSION

TPM may be effectively used for the calculation of effective parameters for a periodic array of cylinders with square cross-section embedded in a matrix material. TPPA gives effective two-sides estimates for homogenized coefficients for the problem under consideration.

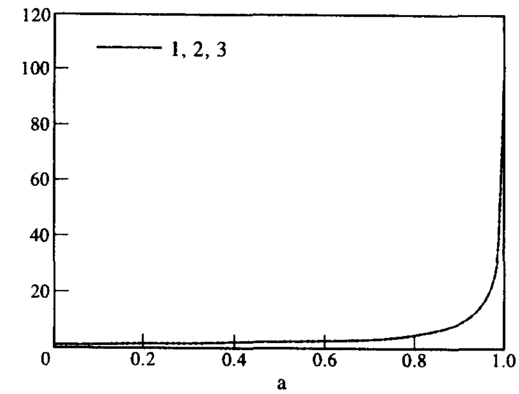

Fig. 5. Dependence of homogenized parameter due to size of inclusions for large inclusion conductivity $\lambda = 100$.

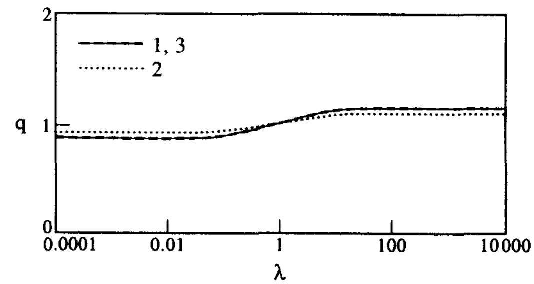

Fig. 6. Dependence of homogenized parameter due to conductivity of inclusions for small inclusion size $\lambda = 0.25$.

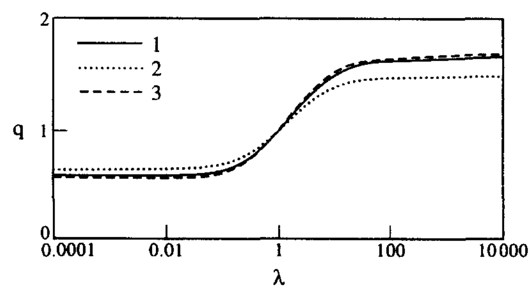

Fig. 7. Dependence of homogenized parameter due to conductivity of inclusions for middle inclusion size $\lambda = 0.5$.

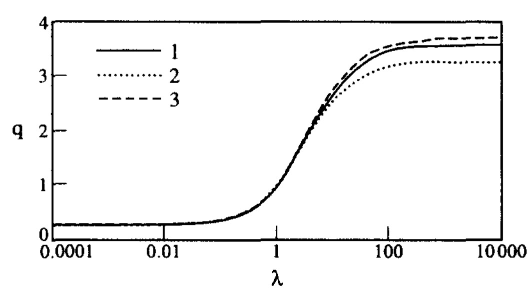

Fig. 8. Dependence of homogenized parameter due to conductivity of inclusions for large inclusion size $\lambda = 0.75$.

REFERENCES

1. Batchelor, G., Transport properties of two-phase materials with random structures. *Annual Review of Fluid Mechanics*, 1974, **6**, 227–255.
2. Maxwell, J. C., *A Treatise on Electricity and Magnetism*. Oxford University Press, Oxford, 1904.
3. Rayleigh, J. W., On the influence of obstacles arranged in rectangular order on the properties of the medium. *Philosophical Magazine Series*, 1892, **5**(34), 481–502.
4. Torquato, S., Random heterogeneous media: microstructure and improved bounds on effective properties. *Applied Mechanics Review*, 1991, **44**, 37–76.
5. Nitham, B. W. and Sammut, R. A., Refractive index of array of spheres and cylinders. *Journal of Theoretical Biology*, 1976, **56**, 125–149.
6. McPhedran, R. C. and McKenzie, D. R., Electrostatic and optical resonances of arrays of cylinders. *Applied Physics*, 1980, **23**, 223–235.
7. Perrinns, W. T., McPhedran, R. C. and McKenzie, D. R., Transport properties of regular arrays of cylinders. *Proceedings of the Royal Society of London*, 1979, **A369**, 207–225.
8. McPhedran, R. C., Poladin, L. and Milton, G. W., Asymptotic studies of closely spaced, highly conducting cylinders. *Proceedings of the Royal Society of London*, 1988, **A415**, 185–196.
9. Kohanenko, Yu., Study of the composites, reinforced by rectangular fibers, by nets method. *Doklady NAN Ukraine, Mathematics, Natural and Technical Sciences*, 1993, **9**, 60–64 (in Russian).
10. Bourgat, J., Numerical experiments of the homogenization method for operators with periodic coefficients. In *Lectures Notes in Maths*, 1979, **704**, 330–356.
11. Baklvalov, N. and Panasenko, G., Averaging Processes in Periodic Media. *Mathematical Problem in Mechanics of Composite Materials*. Kluwer Academic Publishers, Dordrecht, 1989.
12. Kerner, E. H., The elastic and thermoelastic properties of composite media. *Proceedings of the Physics Society*, 1956, **69**, 808–820.
13. Van der Pol, C., On the rheology of concentrated dispersions. *Rheology Acta*, 1958, **1**, 198–211.
14. Christensen, R., *Mechanics of Composite Materials*. Wiley, New York, 1979.
15. May, S., Tokarzewski, S., Zachara, A. and Cichocki, B., Continued fraction representation for the effective thermal conductivity coefficient of a regular two-component composite. *International Journal of Heat and Mass Transfer*, 1994, **37**(4), 2165–2173.
16. Andrianov, I., Blawzdziewicz, J. and Tokarzewski, S., Effective conductivity for densely parked highly condensed cylinders. *Applied Physics A*, 1994, **59**, 601–604.
17. Bensoussan, A., Lions, J.-L. and Papanicolaou, G., *Asymptotic Methods in Periodic Structures*. North-Holland, New York, 1978.
18. Andrianov, I. V., Shevchenko, V. V. and Kholod, E. G., Asymptotic methods in the statics and dynamics of perforated plates and shells with periodic structures. *Technische Mechanik*, 1995, **15**(2), 141–157.
19. Baker, G. and Graves-Morris, P., Pade approximants. Part II. *Extensions and Applications*. Addison-Wesley, London. 1981.
20. Andrianov, I. V. and Starushenko, G. A., Asymptotic methods in the theory of perforated membranes of nonhomogeneous structures. *Engineering Transction*, 1995, **43**(1–2), 5–18.
21. Kantorovich, L. V. and Krylov, V. I., *Approximate Methods of Higher Analysis*. Fizmatgiz, Moscow, 1962.